Systems Analysis: Solutions to the Problems

Last update: 6 Jan 2012

1.1 Model formation

a) System boundary = country border
External relations: immigration
Internal relations: birth rate, death rate, outmigration

b) System boundary = planet Earth
External relations: none
Internal relations: birth rate, death rate

c) System boundary = planetary surface and upper atmosphere
External relations: input from the ocean, mineralization / respiration
of the biosphere, burning of fossil fuels
Internal relations: photosynthesis, transfer into the ocean
(also see Fig. 5.12)

d) System boundary = water surface and boundary to the (non-reactive)
sediment
External relations: Pb content of the sedimenting particles
Internal relations: re-solution of Pb via pore water into the lake,
transfer into the non-reactive (deep) sediment

1.2 Solar system

a) Gravitational forces between the planets and the sun

b) None

c) Gravitational forces between the solar system and other fixed stars
(external relation)

1.3 From chemical elements to nuclear physics

1. Collecting: searching for chemical elements and their distinction from
molecules

2. Organizing: finding elements with related properties, such as noble
gases, alkaline metals, alkaline earth metals, etc.

3. Understanding: Bohr model of atoms, electron shells

4. Generalizing: quantum mechanics, Pauli principle for electrons

5. Predicting: completing the periodic system, transuranic elements

2.1 Mass balance

a) $\dfrac{\mathrm{d}M}{\mathrm{d}t}$ = (input from factory) – (transfer into atmosphere) – (chemical decomposition in the lake) – (outflow)

b) $\dfrac{\mathrm{d}M}{\mathrm{d}t}$ = (transfer into atmosphere) – (sorption and sedimentation) – (chemical decomposition) – (outflow)

Note: the substance mass entering the lake due to the accident is described by the initial condition $M_o = M(t=0)$.

2.2 Determining the dimension of parameters

$$[k_1] = \mathrm{M L^{-3}\ T^{-1}}; [k_2] = \mathrm{L^2\ T^{-1}}; [k_3] = \mathrm{L\ T^{-1}}; [k_4] = \mathrm{M^{-1}\ L^3\ T^{-1}}$$

2.3 Phosphorus sedimentation

$10\ \mathrm{m\ yr^{-1}}$, the mean „sinking velocity" of phosphorus

2.4 Bed of nails

Bernoulli numbers for $n = 8$:

$$\frac{1}{256},\ \frac{8}{256},\ \frac{28}{256},\ \frac{56}{256},\ \frac{70}{256},\ \frac{56}{256},\ \frac{28}{256},\ \frac{8}{256},\ \frac{1}{256}$$

Probability for box $m = 0$: $\dfrac{70}{256} = 0.273$

3.1 Dimensional Henry coefficient

$$[K_H] = \mathrm{L^2 T^{-2}},\ \text{Units e.g. (atm L mol}^{-1}\text{) or (bar m}^3\text{kg}^{-1}\text{)}$$

$$K_{a/w} = \frac{K_H}{RT}\quad (R: \text{gas constant},\ T: \text{absolute temperature})$$

3.2 Henry coefficient of methyl bromide

$$K_H = (6.5 \pm 0.5) \text{ atm L mol}^{-1}$$

$$K_{a/w} = \frac{(6.5 \pm 0.5)}{24.0} = 0.27 \pm 0.02 \qquad \text{(with } R = 0.082 \text{ L atm mol}^{-1} \text{ K}^{-1}, \; T = 293 \text{ K)}$$

3.3 Methyl bromide as ozone killer

Methyl bromide mass = 67.5 mol = 6.4 kg
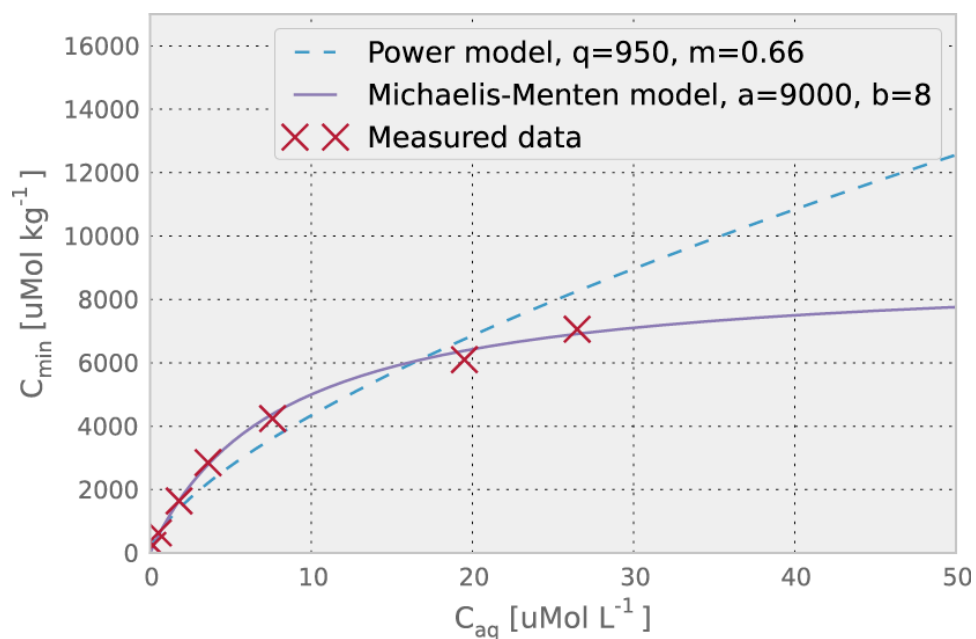
3.4 Nonlinear sorption isotherm

The usual expression for sorption equilibria has the form $C_{min} = q \, (C_{aq})^m$. In double logarithmic form, this corresponds to a linear regression: $\ln(C_{min}) = m \cdot \ln(C_{aq}) + \ln q$.

The model is not entirely satisfactory. For higher concentrations, it systematically overestimates C_{min} (see figure). As alternative, we use the following model (Michaelis-Menten model):

$$C_{min} = a \cdot \frac{C_{aq}}{b + C_{aq}}$$

Here, the parameter a represents the saturation concentration and b the value of C_{aq} at which half of the (sorption) saturation is reached. This second model shows a much better fit to the measured data.
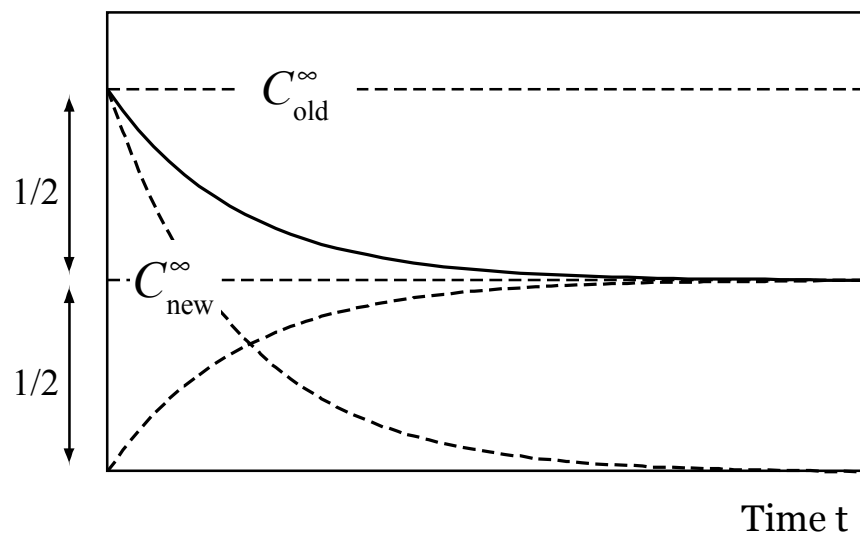
3.5 Economic theory: supply and demand

a) Intercept point of the curves

b) Price rises

c) More machines sold, price sinks

4.1 Adjustment behavior

(as in Fig. 4.3)

4.2 Radioactive decay

a) $N_{Rn}(t) = N_{Rn}^{\infty} + (N_{Rn}^{\circ} - N_{Rn}^{\infty})e^{-\lambda_{Rn}t}$

with $\lambda_{Ra} = 1.187 \times 10^{-6}\,d^{-1}$, $\lambda_{Rn} = 0.1824\ d^{-1}$
$N_{Rn}^{\circ} = 5 \times 10^{4}\,\text{atoms/liter}$

$N_{Rn}^{\infty} = \dfrac{\lambda_{Ra}}{\lambda_{Rn}} N_{Ra} = 650\ \text{atoms/liter}$

b) $A_i = \lambda_i N_i,\ i = \text{Ra, Rn}$
$A_{Rn}(t) = A_{Rn}^{\infty} + \left(A_{Rn}^{\circ} - A_{Rn}^{\infty}\right)e^{-\lambda_{Rn}t}$

with $A_{Rn}^{\infty} = A_{Ra} = 119\ \text{decays/liter/d}$
$A_{Rn}^{\circ} = 9.12 \times 10^{3}\ \text{decays/liter/d}$

4.3 Cars in a parking lot

a) 425 parking spots

b) 100 minutes

c) 3.2 hours

4.4 Phosphorus in a lake with sedimentation

$C^\infty\left(10\ \text{t/yr}\right)=40\ \text{mg m}^{-3}$ (i.e. the current state of the lake corresponds to the steady state of the current input)

$C^\infty\left(6\ \text{t/yr}\right)=24\ \text{mg m}^{-3}$

$\tau_{5\%}=2.4\ \text{yr}$

4.5 Exponential immigration

1975: Immigration $\quad$ 0.25 x 10^6 yr $^{-1}$

2000: Immigration $\quad$ 1.19 x 10^6 yr $^{-1}$

$\qquad$ Population $\qquad$ 35.1 x 10^6

4.6 Lake water temperature

a) $\dfrac{d\Theta}{dt}=(k_\Theta+k_w)(\Theta_{eq}-\Theta)=k(\Theta_{eq}-\Theta)$ $\quad$ with $k=(k_\Theta+k_w)$

b) Annual variation corresponds to $\omega=0.017\,\text{d}^{-1}$
Lake A: $k_A=0.03\,\text{d}^{-1}+0.02\ \text{d}^{-1}=0.05\ \text{d}^{-1}$
$$\eta_A=\arctan\left(\frac{0.017\text{d}^{-1}}{0.05\text{d}^{-1}}\right)=18.8°\quad;\quad \Delta T_A\approx19\,\text{days}$$

c) Lake B with identical thermal properties, without through-flow

Lake B: $k_A=0.03\,\text{d}^{-1}$
$$\eta_A=\arctan\left(\frac{0.017\text{d}^{-1}}{0.03\text{d}^{-1}}\right)=29.5°\quad;\quad \Delta T_A\approx30\,\text{days}$$

d) $\dfrac{\Theta_{amp,A}}{\Theta_{amp}}=\dfrac{k_A}{\sqrt{k_A^2+\omega^2}}=0.947\quad;\quad \dfrac{\Theta_{amp,B}}{\Theta_{amp}}=\dfrac{k_B}{\sqrt{k_B^2+\omega^2}}=0.870$

$$\frac{\Theta_{amp,A}}{\Theta_{amp,B}} = 1.088$$

Lake A is better coupled to the annual temperature forcing and thus has slightly larger temperature amplitudes.

4.7 Dye in a well

a) $\quad \dfrac{dC}{dt} = -(k_r + k_w)C, \quad k_w = 0.06 \text{ h}^{-1}$

$$C_0 = 5 \text{ x } 10^{-4} \text{ g/L}$$

b) $\quad t_{crit} = 24.5 \text{ h}$

c) $\quad \dfrac{dC}{dt} = -k_r C, \quad t_{crit} = 39 \text{ h}$

4.8 Degradation process in a sewage plant

a) $\quad \dfrac{dC}{dt} = k_w C_{in} - k_w C - R \quad, \qquad\qquad k_w = \dfrac{1}{\tau_w} = 0.5 \text{ d}^{-1}$

b) $\quad$ Since measurements show that $C^\infty = \dfrac{1}{20} C_{in}$, it follows that $R = 19 \, k_w C^\infty$

c) $\quad k_r = 19 \, k_w = 9.5 \text{ d}^{-1}$

d) $\quad \tau_{5\%} = \dfrac{3}{k_r + k_w} = 0.15 \text{ d} = 3.6 \text{ h}$

e) $\quad \tau_{5\%} = 0.15 \text{ d} \; (\tau_{5\%}$ does not change, since the system is linear!)

4.9 Copper accumulation on farm land

a) $\quad \dfrac{dm}{dt} = j - km \qquad\qquad m : \text{Cu per m}^2$

$\qquad\qquad\qquad\qquad\qquad\qquad\quad j : \text{ Input per m}^2 \text{ and year}$

$\qquad\qquad\qquad\qquad\qquad\qquad\quad k = 0.006 \text{ yr}^{-1}$

b) $\quad m^\infty = 7000 \text{ mg m}^{-2}$

c) $\quad \tau_{5\%} = 500 \text{ yr}$

d) $k = 0.007 \text{ yr}^{-1}$

e)   No, because input growth rate $\beta = 0.01$ is not $\ll k$

4.10   Tritium in a lake

The relevant parameters have the following values:

Decay constant of tritium: $k_\lambda = 0.058 \text{ yr}^{-1}$

$k_w = 0.1 \text{ yr}^{-1}$

$C_{in} \equiv C_{in}^\circ = 1 \text{ Bq/L}$ (Bq = Becquerel)

$C_{lake} = C^\circ = 0.44 \text{ Bq/L}$

a) $\dfrac{dC}{dt} = k_w C_{in} - (k_w + k_\lambda) C = k_w C_{in} - k_t C, \; k_t = k_w + k_\lambda$

b)   Since $\dfrac{C^\circ}{C_{in}} = 0.44, \; \dfrac{k_w}{k_w + k_\lambda} = 0.63,$ there must be an additional elimination   process.

Assumption: This elimination is first order with

$k_r = k_w \dfrac{C_{in}}{C^\circ} - (k_w + k_\lambda) = 0.069 \text{ yr}^{-1},$ i.e. $k_t = k_w + k_\lambda + k_r = 0.227 \text{ yr}^{-1}$

c) $C_{in}(t) = C_{in}^\circ e^{\beta t}$ with $\beta = 0.02 \text{ yr}^{-1}$

$C(t = 10\text{yr}) = 0.50 \text{ Bq/L}, \; C^\infty(t = 10\text{yr}) = 0.54 \text{ Bq/L}$

→ Perturbation by increasing input is nearly adiabatic.

d)   Like c), but with $\beta = \dfrac{1}{1 \text{ yr}} \ln(1.2) = 0.182 \text{ yr}^{-1}$

$C(t = 10\text{yr}) = 1.53 \text{ Bq/L}, \; C^\infty(t = 10\text{yr}) = 2.72 \text{ Bq/L}$

→ strongly non-adiabatic disturbance

5.1 Reactor with two substances

For parameters from Examples 5.2 and 5.3, in particular
$k_w = 0.04\ \text{h}^{-1}$, t in $\left[\text{h}^{-1}\right]$:

$$C_1(t)\left[\text{mol m}^{-3}\right] = 0.091 - 0.0196\ \text{e}^{-0.04t} - 0.0714\ \text{e}^{-0.55t}$$

$$C_2(t)\left[\text{mol m}^{-3}\right] = 0.909 - 0.9804\ \text{e}^{-0.04t} + 0.0714\ \text{e}^{-0.55t}$$

5.2 Chain of lakes

a)
$$\frac{\text{d}M_1}{\text{d}t} = J_1 - \frac{Q_{1,2}}{V_1}M_1 - k_\lambda M_1$$

$$\frac{\text{d}M_2}{\text{d}t} = J_2 + \frac{Q_{1,2}}{V_1}M_1 - \frac{Q_2}{V_2}M_2 - k_\lambda M_2$$

This results in an identical coefficient matrix as in Eq. (5.34).

b)
$$\frac{\text{d}M_{\text{tot}}}{\text{d}t} = J_1 + J_2 - k_\lambda M_{\text{tot}} - Q_2 C_2$$

The additional variable C_2 would be proportional to M_{tot} and could by replaced provided that (1) $J_2 = 0$; and (2) the system is at (quasi) steady state. Then the following holds:

$$C_2 = \frac{M_{\text{tot}}}{V_2\left[1 + \dfrac{V_1}{Q_{1,2}}\left(\dfrac{Q_2}{V_2} + k_\lambda\right)\right]}$$

c) Only the weighted sum of the concentration, $\overline{C} = \dfrac{C_1 V_1 + C_2 V_2}{V_1 + V_2}$ satisfies

conservation of mass.

5.3 Radioactive decay chain with three isotopes

$$\frac{\text{d}A_X}{\text{d}t} = -\lambda_X A_X \qquad\qquad \lambda_X = 0.182\text{d}^{-1} = 7.6\text{x}10^{-3}\ \text{h}^{-1}$$

$$\frac{\text{d}A_Y}{\text{d}t} = \lambda_Y\left(A_X - A_Y\right) \qquad\qquad \lambda_Y = 1.55\ \text{h}^{-1}$$

$$\frac{\text{d}A_Z}{\text{d}t} = \lambda_Z\left(A_Y - A_Z\right) \qquad\qquad \lambda_Z = 2.10\ \text{h}^{-1}$$

Since

$\lambda_Y, \lambda_Z\ ?\ \lambda_X$, for times t with $t\lambda_Y\ ?\ 1$ or $t\lambda_Z\ ?\ 1$, the following approximation holds :
$$A_Y(t):\ A_Z(t):\ A_X(t) = A_X(0)\text{e}^{-\lambda_X t}$$

5.4   Tritium in a sewage treatment plant

a) $\dfrac{\mathrm{d}A_1}{\mathrm{d}t} = -\left(k_{F1} + k_w + k_\lambda\right)A_1 + k_{F1}A_2$

$\dfrac{\mathrm{d}A_2}{\mathrm{d}t} = k_{F2}A_1 - k_{F2}A_2$

$k_\lambda = \dfrac{\ln 2}{\tau_{1/2}} = 1.58 \text{ x } 10^{-4}\,\mathrm{d}^{-1},\ k_w = \dfrac{Q}{V_1} = 1\,\mathrm{d}^{-1}$

$k_{F1} = \dfrac{Q_F}{V_1} = 6 \text{ x } 10^{-3}\,\mathrm{d}^{-1},\ k_{F2} = \dfrac{Q_F}{V_2} = 0.02\,\mathrm{d}^{-1}$

b)   Since k_λ is much small than other specific rates, decay plays no role

c) $\lambda_1:\ -k_w,\ \ \lambda_2:\ -k_{F2}$

d) $A_1(t) = \text{const. } \exp\left(-k_{F2}t\right)$

5.5   Stratified lake with sedimentation

a)   In $V_1:\ \tau_w = 100$ d; $\tau_{ex,1} = 20$ d; $\tau_{tot} = 16.7$ d
      In $V_2:\ \tau_{ex,2} = 40$ d

b   through d)

	b) conservative	c) radioactive	d) conservative + sedimentating
Steady-state conc'ns: $C_1^\infty\left[\mathrm{mg\ m^{-3}}\right]$	100	29	46
C_2^∞	100	21	54
Eigenvalues $\lambda_1\left[\mathrm{d}^{-1}\right]$	-0.00305	-0.0131	-0.0063
$\lambda_2\left[\mathrm{d}^{-1}\right]$	-0.0819	-0.0919	-0.104
$\tau_{5\%}:\ \dfrac{3}{\lvert\lambda_1\rvert}[\mathrm{d}]$	980	230	480

e)   Note that the smaller sedimentation rate in the deep water leads to a slightly increased concentration in the volume V_2 (case d).

5.6   Conservative substance in a chain of lakes

a) $C_1(t) = C_1^\circ e^{-k_{w1}t}$

$C_2(t) = C_1^\circ \dfrac{k_{w2}}{k_{w1} - k_{w2}} \left(e^{-k_{w2}t} - e^{-k_{w1}t} \right)$

with $k_{w1} = \dfrac{Q_\circ}{V_1} = 1 \text{ d}^{-1}$, $k_{w2} = \dfrac{Q_\circ}{V_2} = 0.05 \text{ d}^{-1}$, $C_1^\circ = \dfrac{M}{V_1} = 2 \text{ gm}^{-3}$

Adjustment times: $t_1 = 5.3 \text{ d}$, $t_2 = 46 \text{ d}$

b) $C_2^{\max} = 0.085 \text{ gm}^{-3}$, reached at time $t_{\max} = 3.2 \text{ d}$

c) $\alpha: k_{w2} = 0.05 \text{ d}^{-1}$

d)   Eigenvalues of the system with pumping:
$\lambda_1 = -5.2 \text{ d}^{-1}$
$\lambda_2 = -0.048 \text{ d}^{-1}$

As a result of the high pumping power Q_P the system $(V_1 + V_2)$ can approximately be described as completely mixed, with the outflow rate

$\dfrac{Q_\circ}{V_1 + V_2} = k_{tot} = 0.0476 \text{ d}^{-1} \sim -\lambda_2$

and initial concentration $C_{tot}^\circ = \dfrac{M}{V_1 + V_2} = 0.095 \text{ gm}^{-3} = 95 \text{ mg m}^{-3}$

The critical concentration
$C_{krit} = 10 \ \mu\text{g/L} = 10 \text{ mg m}^{-3}$ is underrun after the time $t_{krit} = 47 \text{ d}$.

e)   See d).

5.7   Temperature control by a thermostat

a) $\dfrac{dy}{dt} = -L + k_1 x$; $\dfrac{dx}{dt} = k_2(y_o - y)$

Purely imaginary eigenvalues $\lambda_i = \pm i (k_1 k_2)^{1/2}$

steady state at $y^\infty = y_o$, $x^\infty = L/k_1$ is not reached (undamped oscillations)

b)   If equations are altered such that x cannot drop below 0, the system becomes nonlinear.

c) k_1 and k_2 should be large.

d) The modification introduced in b) would dampen the oscillations. Alternatively, the 'valve equation' could be modified such that at large valve positions x the opening process is slowed down.

$$\frac{dx}{dt} = k_2(y_o - y) - k_\varepsilon x$$

6.1 Mountain lake with time-dependent elimination rate

a) System is linear

b) No

c) $j_{in}(t)$ und $k(t)$ are external relations

d) Integration piece-wise for periods with $k = \text{const.}$, whereby the terminal value becomes the initial value of the next period.
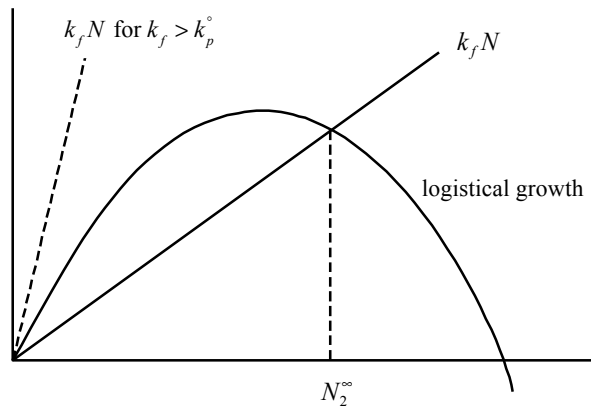
6.2 Fish in a pond

Note: The problem refers to Example 6.2 in Section 6.1.3.

a) $$\frac{dN}{dt} = k_p^\circ N\left(1 - \frac{N}{N_{max}}\right) - k_f N$$

Fixed point: $N_1^\infty = 0, \quad N_2^\infty = \dfrac{k_p^\circ - k_f}{k_p^\circ} N_{max}$

If $k_p^\circ > k_f$ (see question d), N_1^∞ is unstable, N_2^∞ stable.

b)

c) $k_f = \dfrac{1}{2}k_p^\circ$. Then, the yield is $Y = k_f N_2^\infty = k_f \dfrac{N_{max}}{2}$

d) $k_p^\circ > k_f$

6.3　Jacobian matrix of the Lotka-Volterra model

$$B(X,Y)=\begin{pmatrix}(k_1-k_3Y) & -k_3X \\ k_3Y & (-k_2+k_3X)\end{pmatrix}$$

$$B(X_1^\infty=0,\ Y_1^\infty=0)=\begin{pmatrix}k_1 & 0 \\ 0 & -k_2\end{pmatrix}$$

The fixed point is a saddle point (Fig. 6.11c) with stability along the Y axis, instability along the X axis.

$$B\left(X_2^\infty=\frac{k_2}{k_3},\ Y_2^\infty=\frac{k_1}{k_3}\right)=\begin{pmatrix}0 & -k_2 \\ k_1 & 0\end{pmatrix}$$

Eigenvalues $\lambda_j=\pm i(k_1k_2)^{1/2}\ \rightarrow$ center (Fig. 6.11f)

Note: It can be shown that the model exhibits an undamped oscillation (see Fig. 6.19c).

6.4　Predator-prey model with self-interaction

Jacobian matrix $B(X,Y)=\begin{pmatrix}(k_1-k_3Y-2k_4X) & -k_3X \\ k_3Y & (-k_2+k_3X)\end{pmatrix}$

At the fixed point $X_2^\infty=\dfrac{k_2}{k_3}$, $Y_2^\infty=\dfrac{k_1}{k_3}-\dfrac{\varepsilon}{k_3}$ with $\varepsilon=\dfrac{k_2k_4}{k_3}$:

$$B(X_2^\infty,Y_2^\infty)=\begin{pmatrix}-\varepsilon & -k_2 \\ (k_1-\varepsilon) & 0\end{pmatrix}$$

Eigenvalues $\lambda_j=\dfrac{1}{2}\left[-\varepsilon\pm\left\{\varepsilon^2+4\varepsilon k_2-4k_1k_2\right\}^{1/2}\right]$

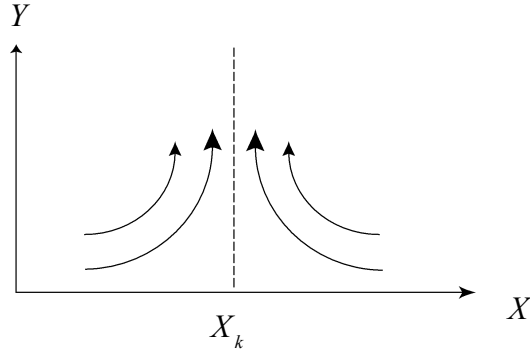If k_4 is so small that $k_1k_2>\dfrac{\varepsilon^2}{4}+\varepsilon k_2$, then the eigenvalues are conjugate-complex with negative real part (case d in Fig. 6.11, also see Fig. 5.11a).

6.5   Semi-trivial fixed point of the Holling-Tanner model

Jacobian matrix $\mathbf{B}\left(X_k,\ Y=0\right)=\begin{pmatrix} -r & -w\dfrac{X_k}{X_k+K_B} \\ 0 & s \end{pmatrix}$

Eigenvalues: $\lambda_1=-r,\ \ \lambda_2=s$ $\qquad\qquad$ $(r,s>0)$

$\rightarrow$ Case c) in Fig. 6.11 (saddle point)

6.6   Nonlinear biomass growth in a pond

Note that f_w has the dimension $\left[(\text{concentration})^{-2}\text{T}^{-1}\right]$.

The solution to this problem becomes clearer if we replace the variable pair $(N,\ B)$ by $\left(M=N+B,\ N\right)$.

c)   If condition (I) of answer d) is met, then the system has three fixed points:

(A) $M_A^\infty=\dfrac{J_N}{Q},\qquad N_A^\infty=\dfrac{J_N}{Q}\qquad\left(i.e.\ B_A^\infty=0\right)$

(B) $M_B^\infty=M_A^\infty=\dfrac{J_N}{Q};\qquad N_B^\infty=\dfrac{1}{2}\left(N_0-N_{crit}\right)$

(C) $M_C^\infty=M_A^\infty=\dfrac{J_N}{Q};\qquad N_C^\infty=\dfrac{1}{2}\left(N_0+N_{crit}\right)$

$\qquad$ with $N_{crit}=\left(N_0^2-4\dfrac{k_a+k_q}{f_w}\right)^{1/2},\qquad k_q=\dfrac{Q}{V}$

$\qquad$ stability $\rightarrow$ see d)

d) **Condition I:** biomass B can only be created if

$$N_0 \geq 2\left(\frac{k_a + k_q}{k_w}\right)^{1/2}, \quad \text{i.e. } N_{crit} \text{ is real}$$

Fixed point (A) is stable for $\dfrac{J_N}{Q} < N_B^\infty$ and $\dfrac{J_N}{Q} > N_C^\infty$

Fixed point (C) is stable for $N_B^\infty \leq \dfrac{J_N}{Q} \leq N_C^\infty$

Fixed point (B) is unstable.

6.7 Lotka-Volterra with two prey animals

a) $\dfrac{dX_1}{dt} = k_1 X_1 - k_4 X_1 Y$

$\dfrac{dX_2}{dt} = k_2 X_2 - k_5 X_2 Y$

$\dfrac{dY}{dt} = k_4 X_1 Y + k_5 X_2 Y - k_3 Y$

b) 3 fixed points
   (I) Trivial fixed point $X_1 = X_2 = Y = 0$
        stable with respect to Y, unstable with respect to X_1 and X_2

   (II) $X_1^\infty = \dfrac{k_3}{k_4}, \qquad X_2^\infty = 0, \qquad Y^\infty = \dfrac{k_1}{k_4}$

         Center if $\dfrac{k_1}{k_4} > \dfrac{k_2}{k_5}$, otherwise unstable

   (III) $X_1^\infty = 0, \qquad X_2^\infty = \dfrac{k_3}{k_5}, \qquad Y^\infty = \dfrac{k_2}{k_5}$

         Center if $\dfrac{k_1}{k_4} < \dfrac{k_2}{k_5}$, otherwise unstable

c) Species with a higher growth/predation coefficient

   $\left(\dfrac{k_1}{k_4} \text{ for } X_1, \dfrac{k_2}{k_5} \text{ for } X_2\right)$ survives. In our example, X_2 survives

   although X_1 has a higher growth rate $(k_1 > k_2)$.
   Predation pressure on X_2 is smaller than on X_1 $(k_5 < k_4)$.

6.8   **Lotka-Volterra with prey niche**

a) $\dfrac{\mathrm{d}X_1}{\mathrm{d}t} = \left(k_1 - k_4\right)X_1 + k_5 X_2 - k_3 X_1 Y$

$\dfrac{\mathrm{d}X_2}{\mathrm{d}t} = \left(k_1 - k_5\right)X_2 + k_4 X_1$

$\dfrac{\mathrm{d}Y}{\mathrm{d}t} = k_3 X_1 Y - k_2 Y$

b)   For $k_1 < k_5$ the following non-trivial fixed point exists:

$X_1^{\infty} = \dfrac{k_2}{k_3}, \qquad X_2^{\infty} = \dfrac{k_2 k_4}{k_3\left(k_5 - k_1\right)}$

$Y^{\infty} = \dfrac{k_1\left(k_5 - k_1 + k_4\right)}{k_3\left(k_5 - k_1\right)}$

c)   If $k_1 > k_5$, the prey in the niche $\left(B_2\right)$ could increase unchecked, and then (by migrating out of the niche) help the predator achieve infinite growth. We can "fix" this behavior by replacing the linear prey growth by a logistical function:

$i = 1,2 \qquad k_1 X_i \rightarrow k_1^{*}\dfrac{X_i}{X_i + X_{krit}}$

6.9   **Competition for living space**

a) $\dfrac{\mathrm{d}A}{\mathrm{d}t} = k_1 A - k_2 AB \equiv f_A(A,B)$

$\dfrac{\mathrm{d}B}{\mathrm{d}t} = (k_3 - k_4 B)B - k_5 AB \equiv f_B(A,B)$

b)   Fixed points:
(1) $A_1^{\infty} = 0 \quad ; \quad B_1^{\infty} = 0$

(2) $A_2^{\infty} = \dfrac{k_3}{k_5} - \dfrac{k_4 k_1}{k_2 k_5} = 2 \quad ; \quad B_2^{\infty} = \dfrac{k_1}{k_2} = 1$

(3) $A_3^{\infty} = 0 \quad ; \quad B_3^{\infty} = \dfrac{k_3}{k_4}$

c) Stability of fixed point 2 (A and B non zero).

$$\text{Jacobian matrix } B(A, B) = \begin{pmatrix} 0 & \dfrac{1}{k_5}(k_2 k_3 - k_1 k_4) \\ -k_5 \dfrac{k_1}{k_2} & -\dfrac{k_1 k_4}{k_2} \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -4 & -4 \end{pmatrix}$$

Eigenvalues: $\lambda_{1,2} = -2 \pm 2i$ (damped oscillations)

7.1 Elimination of the inhomogeneous term

1-dimensional: With $\widehat{v}^{(n)} = v^{(n)} + \dfrac{I}{a_0 - 1}$ it follows that $\widehat{v}^{(n+1)} = a_0 \widehat{v}^{(n)}$

q-th order: prove Eq. (7.17).

7.2 Consumer loan

$(1.025)^{12} = 1.345 \qquad \rightarrow \qquad$ annual interest 34.5%

7.3 Reactor with chloride

a) 50 kg

b) 14 weeks

7.4 Returnable bottles

a) Use the method of the characteristic equation.

b) Because the system is linear, we can choose a different approach for the first year and simply combine that with the original result, whereby the index n is moved by a month:

$N^{(n)} = N_1^{(n)} + N_2^{(n)}$ with

$N_1^{(n)} = 5{,}300\left[(\lambda_1)^n - (\lambda_2)^n\right]$

$N_2^{(n)} = 50{,}000 - 48{,}850(\lambda_1)^{n-1} - 1{,}150(\lambda_2)^{n-1}; \quad n \geq 1$

$\lambda_1 = 0.822, \qquad \lambda_2 = -0.122$

7.5   **Fibonacci numbers**

$$y^{(n)} = \frac{1}{\sqrt{5}}\left[\left(\lambda_1\right)^n - \left(\lambda_2\right)^n\right]$$

with $\lambda_1 = \frac{1}{2}\left(1+\sqrt{5}\right);\quad \lambda_2 = \frac{1}{2}\left(1-\sqrt{5}\right)$

7.6   **Gambling**

a) $Y^{(0)} > 2X^{(0)}$

b) $Y^{(0)} = 2X^{(0)}$

7.7   **Students in a study program**

550

7.8   **Fish in a pond**

b) $N_1^{(i)} = 10{,}000 + 0.5\,N_1^{(i-1)} + 2\,N_2^{(i-1)}$
$N_2^{(i)} = 0.1\,N_1^{(i-1)}$

c)   Use $N_2^{(i-1)} = 0.1\,N_1^{(i-2)}$
$\rightarrow 10{,}000 - N_1^{(i)} + 0.5\,N_1^{(i-1)} + 0.2\,N_1^{(i-2)} = 0$

d) $N_1^{(1)} = 10{,}000;\quad N_1^{(2)} = 15{,}000;\quad N_1^{(3)} = 19{,}500$
$N_1^{(4)} = 22{,}750;\quad N_1^{(5)} = 25{,}275$

e) $\widehat{N}_1^{(i)} = N_1^{(i)} - 33{,}333.3$ makes the equation homogeneous.

Characteristic exponents:
$$\lambda_{1,2} = \frac{1}{2}\left(0.5 \pm \sqrt{1.05}\right)$$
Initial values: note that $N_2^{(0)} = 0$ corresponds to $N_1^{(-1)} = 0$.

$$N_1^{(i)} = 33{,}333.3 - 31{,}305.3\left(\lambda_1\right)^i - 2{,}028.0\left(\lambda_2\right)^i$$

f) $N_1^{(\infty)} = 33{,}333.3$

$N_2^{(\infty)} = 3{,}333.3$

8.1 Tetrachloroethylene in a pond

a) 69 d

b) 34.5 d

8.2 Oxygen in the lake

This problem relates to Example 8.2 in Section 8.2.5.

Initially, C_A increases slightly (steady state concentration 12.8 mg/L) but as the deep water becomes anoxic, decreases to the slightly lower steady state value of 11.7 mg/L.

8.3 Geothermal heat flux at the bottom of a lake

a) With $\dfrac{dT}{dx} = 0.175 \text{ km}^{-1}$ (x: Depth in the sediment, positive downwards)

we get $F_{th} = -0.10 \text{ Wm}^{-2}$ (Flux, positive upwards)

(Caution: the correct unit for γ_{th} is $\text{Wm}^{-1}\text{K}^{-1}$)

b) $\dfrac{dT}{dz} = +4.8 \text{ x } 10^{-3} \text{km}^{-1}$ (temperature increases with water depth z.)

8.4 Vertical turbulent diffusivity

a) $K_z = 2.0 \text{ m}^2\text{d}^{-1} = 2.32 \text{ x } 10^{-5}\text{m}^2\text{s}^{-1}$

b) $F_{Rn} = 3.0 \text{ x } 10^4 \text{ Bq m}^{-2}\text{d}^{-1}$
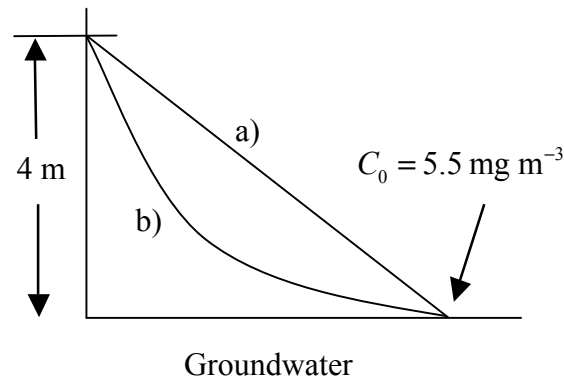
8.5 Péclet und Damköhler numbers

a) $\dfrac{\tau_{diff}}{\tau_{ad}} = \dfrac{x_L^2/2D}{x_L/v} = \dfrac{1}{2}\dfrac{x_L v}{D} = \dfrac{1}{2}\text{Pe}$

b) $\left(\dfrac{x_{diff}}{x_{ad}}\right)^2 = \dfrac{2D/k_r}{\left(v/k_r\right)^2} = 2\dfrac{Dk_r}{v^2} = 2\ \text{Da}$

(the factor 2 plays no role for this type of consideration!)

8.6 Volatile substance in the groundwater

a) + b)

Groundwater

c) $F = 1.4 \times 10^{-3}\ \text{mg m}^{-3}\text{d}^{-1}$

d) $\tau_{diff} = 1143\ \text{d} = 3.1\ \text{yr}$

e) $\text{Pe} = 57,$ that is, transport flux is dominated by advection

$\tau_{adv} = 40\ \text{d}$

$F_{adv} = 0.15 \cdot v\ C_0 = 0.083\ \text{mg m}^{-3}\text{d}^{-1}$

8.7 Transport and reaction in a lake's water column

a) Chemical A = Curve 4; B = 2; C = 3; D = 1

b) Vertical transport equation of chemical A and D. For chemical A, the first-order reaction rate $k_r = 0$. z is vertical coordinate, $z = 0$ at water surface.

$\dfrac{\partial C}{\partial t} = -k_r C + D\dfrac{d^2C}{dz^2} = 0$

Solution for chemical A:

$C(z) = C_0 + az.$ Since $\left.\dfrac{\partial C}{\partial z}\right|_{z=z_b} = 0$ (no flux at sediment!), we get $a = 0$.

For chemical D:

$$C(z) = p\, e^{\lambda z} + q\, e^{-\lambda z}, \qquad \lambda = \left(\frac{k_r}{D}\right)^{1/2}$$

From the boundary conditions $C(z=0) = C_0$ and

$\left.\dfrac{\partial C}{\partial z}\right|_{z=z_b} = 0$, the coefficients p and q are:

$$p = C_0\, \frac{e^{-\lambda z_b}}{e^{\lambda z_b} + e^{-\lambda z_b}} \quad ; \quad q = C_0\, \frac{e^{\lambda z_b}}{e^{\lambda z_b} + e^{-\lambda z_b}}$$

Inserting p and q yields:

$$C(z) = C_0\, \frac{\cosh\left[\lambda\left(z - z_b\right)\right]}{\cosh\left[\lambda z\right]}$$

c)   For fast reaction ($k_r \to \infty$), the concentration in those layers in which reaction occurs drops sharply. Thus, at the sediment surface the curves for the reactive chemicals (B,C,D) are virtually zero. The concentration of chemical B drops linearly through the thermocline to virtually zero at the oxic/anoxic interface. Chemical C drops sharply from C_O at the surface to virtually zero and remains small in the rest of the water column. D has the same profile as C, since nothing will be left for reaction in the deep anoxic layer.

8.8   **Radium and tritium in the ocean deep water**

	F_{deep}^{ad}	F_{deep}^{diff}	F_{sur}^{ad}	F_{sur}^{diff}	Decay	Sum
Radium	2.00	0.61	0	−1.60 0	−1.01	0
Tritium	0	0	−2.0	14.7	−12.7	0

Note: the numbers in the table are generalized transfer velocities in $\left[\mathrm{m\,yr^{-1}}\right]$. If we multiply them by the corresponding boundary values (C_{BW} for radium, C_{OW} for tritium, in $\left[\mathrm{Bq\,m^{-3}}\right]$), we get the fluxes in $\left[\mathrm{Bq\,m^{-2}a^{-1}}\right]$.

8.9   **Time-dependent diffusion/advection equation**

Note: $\dfrac{\partial}{\partial t}C\left(x-v_x t\right)=-v_x C'$ with $C'=\dfrac{dC\left(\xi\right)}{d\xi}$

$$\frac{\partial}{\partial x}C\left(x-v_x t\right)=C'$$

8.10   **Symmetrical diffusion at a boundary surface**

$$C\left(x,t\right)=\frac{C_0}{2}\,\mathrm{erfc}\left(\frac{x}{2\left(D_x t\right)^{1/2}}\right),\ \text{boundary at } x=0$$

Note: $C\left(x=0,t\right)=\dfrac{C_0}{2}$ (for all times $t>0$